

A NOTE ON THE ORDER OF THE SCHUR MULTIPLIER OF p -GROUPS

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ABSTRACT. Let G be a finite p -group of order p^n with $|G'| = p^k$. Let $M(G)$ denotes the Schur multiplier of G . A classical result of Green states that $|M(G)| \leq p^{\frac{1}{2}n(n-1)}$. In 2009, Niroomand, improving Green's and other bounds on $|M(G)|$ for a non-abelian p -group G , proved that $|M(G)| \leq p^{\frac{1}{2}(n-k-1)(n+k-2)+1}$. In this article we note that a bound, obtained earlier, by Ellis and Weigold is more general than the bound of Niroomand. We derive from the bound of Ellis and Weigold that $|M(G)| \leq p^{\frac{1}{2}(d(G)-1)(n+k-2)+1}$ for a non-abelian p -group G . Moreover, we sharpen the bound of Ellis and Weigold and as a consequence derive that if G^{ab} is not homocyclic then $|M(G)| \leq p^{\frac{1}{2}(d(G)-1)(n+k-3)+1}$. We further note an improvement in an old bound given by Vermani. Finally we note, for a p -group of coclass r , that $|M(G)| \leq p^{\frac{1}{2}(r^2-r)+kr+1}$. This improves a bound by Moravec.

1. INTRODUCTION

Let G be a group. The center and the commutator subgroup of G are denoted by $Z(G)$, and $\gamma_2(G)$ respectively. By $d(G)$ we denote the minimal no of generators of G . We write $\gamma_i(G)$ and $Z_i(G)$ for the i -th term in the lower and upper central series of G respectively. Finally, the abelianization of the group G , i.e. $G/\gamma_2(G)$, is denoted by G^{ab} .

Let G be finite p -group of order p^n and let $M(G)$ denotes the Schur multiplier of G . In 1956 Green proved that $|M(G)| \leq p^{\frac{1}{2}n(n-1)}$ [3]. Since then Green's bound has been reproved and generalized by many mathematicians. Weigold, in 1965, gave a bound on $|\gamma_2(G)|$ in terms of $|G/Z(G)|$ and rederived the Green's bound using the existence of representation groups [12]. In 1967 Gaschütz et al., sharpening Green's bound, proved in [2] that

$$|M(G)| \leq |M(G^{ab})| |\gamma_2(G)|^{d(G/Z(G))-1}.$$

The bound of Gaschütz et al. was further generalized by Vermani in 1969 [10]. He obtained their result as a corollary of the bound

$$|M(G)| \leq \left| M\left(\frac{G}{\gamma_c(G)}\right) \right| \left| \text{Hom}\left(\frac{G}{Z_{c-1}(G)}, \gamma_c(G)\right) \right| / |\gamma_c(G)|,$$

where c is the nilpotency class of G . This bound was reproved by Jones using a different method in [5]. In 1969 Green's bound was generalized by Weigold [13] when he proved that

$$(1.1) \quad |M(G)| \leq p^{\frac{1}{2}(d(G)-1)(2n-d(G))}.$$

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In 1972 Jones, generalizing Green's bound, proved in [4] that, if the exponent of the center is $p^{e_{Z(G)}}$, then

$$(1.2) \quad |\gamma_2(G)||M(G)| \leq p^{\frac{1}{2}(n-e_{Z(G)})(n+e_{Z(G)}-1)}.$$

This bound of Jones was further generalised by Vermani in 1974 [11]. He proved that if the restriction homomorphism from $M(G)$ to $M(K)$, for a central subgroup K , is zero (Note that for a cyclic central subgroup K , it is zero), then

$$(1.3) \quad |\gamma_2(G)||M(G)| \leq |(G/K)^{ab} \otimes K| p^{\frac{1}{2}(m-r)(m+r-1)},$$

where m and r are given by $|G/K| = p^m$ and $|\gamma_2(G)K/K| = p^r$.

In 1999 Ellis and Weigold, sharpening Weigold's earlier bound 1.1, proved that

$$(1.4) \quad |M(G)| \leq p^{\frac{1}{2}(d(G)-1)(2n-m)} (= p^{\frac{1}{2}(d(G)-1)(n+k)}),$$

where m and k are given by $|G^{ab}| = p^m$, $|\gamma_2(G)| = p^k$ [1].

Using the bound 1.2 Jones derived the corollary that $|M(G)| \leq p^{\frac{1}{2}n(n-1)-k}$. Vermani, using the bound of Gaschütz et al., noticed in [11, Proposition 2.2] that

$$|M(G)| \leq p^{\frac{1}{2}(n-k-1)(n+k)} = p^{\frac{1}{2}n(n-1)-\frac{1}{2}k(k-1)}.$$

For non-abelian p -groups G , Niroomand proved that

$$(1.5) \quad |M(G)| \leq p^{\frac{1}{2}(n-k-1)(n+k-2)+1},$$

where k is given by $|\gamma_2(G)| = p^k$ [9]. Further, in another paper Niroomand and Russo proved that the bound 1.5 is better than the bound 1.4 of Ellis and Weigold provided G^{ab} is elementary abelian [8, Theorem 1.2].

We mention here that the bound 1.4 of Ellis and Weigold was derived from the following more general bound of theirs.

$$(1.6) \quad |M(G)| \leq p^{\frac{1}{2}d(m-e)+(\delta-1)(n-m)-\max(0,\delta-2)}$$

where m and p^e are the order and the exponent of G^{ab} respectively and d and δ are the minimal no. of generators of G and $G/Z(G)$ respectively.

We then note that the bound 1.6 of Ellis and Weigold is more general than the bound 1.5 of Niroomand. In the following theorem we see that a visibly more general bound than the bound 1.5 can be derived from the bound. 1.6.

Theorem 1.1. *Let G be a non-abelian finite p -group of order p^n with $|\gamma_2(G)| = p^k$ and $d(G) = d$. Then*

$$|M(G)| \leq p^{\frac{1}{2}(d-1)(n+k-2)+1}.$$

Ellis and Weigold also noticed that their bound 1.4 is attained if $G = C_{p^e} \times C_{p^e} \times \dots \times C_{p^e}$. The bound 1.4 of Ellis and Weigold was rederived by Niroomand and Russo. They further improved the bound when $G^{ab} \neq C_{p^e} \times C_{p^e} \times \dots \times C_{p^e}$ proving that

$$(1.7) \quad |M(G)| \leq p^{\frac{1}{2}(d(G)-1)(n+k-1)}$$

in this case.

Now of course Theorem 1.1 provides a visibly stronger bound than the bound 1.7. But the bound 1.7 motivates us to investigate further the case $G^{ab} \neq C_{p^e} \times C_{p^e} \times \dots \times C_{p^e}$. The following theorem sharpens the bound 1.6.

Theorem 1.2. *Let G be a finite p -group of order p^n with $d(G) = d$, $d(G/Z(G)) = \delta$ and $G^{ab} = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \dots \times C_{p^{\alpha_d}} (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d)$. Then*

$$|M(G)| \leq p^{\frac{1}{2}(d-1)(n-k-(\alpha_1-\alpha_d))+(\delta-1)k-\max(0,\delta-2)}.$$

To see that the bound in the above corollary is better than the bound 1.6, we divide the right hand side of the bound 1.6 by the right hand side of the above bound and get the value $p^{\frac{1}{2}[(d-1)(\alpha_1-\alpha_d)-d\alpha_1+n-k]}$ which equals $p^{\frac{1}{2}[-(d-1)\alpha_d-\alpha_1+n-k]}$. Which on putting $n-k = \alpha_1 + \alpha_2 + \dots + \alpha_d$ becomes $p^{\frac{1}{2}[\alpha_2+\dots+\alpha_d-(d-1)\alpha_d]}$. But this value is clearly greater than or equal to 1 because $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d$.

As a consequence we derive the following corollary.

Corollary 1.3. *Let G be a non-abelian finite p -group of order p^n with $d(G) = d$, $|\gamma_2(G)| = p^k$ and $G^{ab} = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \dots \times C_{p^{\alpha_d}} (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d)$. Then*

$$|M(G)| \leq p^{\frac{1}{2}(d-1)(n+k-2-(\alpha_1-\alpha_d))+1}.$$

In particular if G^{ab} is not homocyclic, then

$$|M(G)| \leq p^{\frac{1}{2}(d-1)(n+k-3)+1}.$$

The bound 1.3 of Vermani comes with a hypothesis, so it can not be compared in general with the bound obtained in Theorem 1.1. Though using Theorem 1.1 we can improve the bound of Vermani for non-abelian finite p -groups of nilpotency class at least 3.

Theorem 1.4. *Let G be a finite p -group of nilpotency class at least 3 and K a central subgroup of G such that the restriction homomorphism from $M(G)$ to $M(K)$ is zero. Also assume that $|G/K| = p^m$ and $|\gamma_2(G)K/K| = p^r$. Then*

$$|\gamma_2(G)||M(G)| \leq |(G/K)^{ab} \otimes K| p^{\frac{1}{2}d(G/K)(m+r-2)+1}.$$

In particular,

$$|\gamma_2(G)||M(G)| \leq |(G/K)^{ab} \otimes K| p^{\frac{1}{2}(m-r)(m+r-2)+1}.$$

By the coclass of a p -group G of order p^n we mean the number $n-c$ where c is the nilpotency class of G . In 2009, Moravec proved, for finite p -group G of coclass r , that $|M(G)| \leq p^{r^2+(k+2)r}$ where k is given by $|\gamma_2(G)| = p^k$ [7, Theorem 1.1]. The following Theorem improves this bound.

Theorem 1.5. *Let G be a finite p -group of order p^n and coclass r with $|\gamma_2(G)| = p^k$. Then $|M(G)| \leq p^{\frac{1}{2}(r^2-r)+kr+1}$.*

2. PROOFS OF THEOREMS

Proof of Theorem 1.1 Since G is non-abelian we have $d(G/Z(G)) = \delta \geq 2$, otherwise $G/Z(G)$ is cyclic and G is abelian. Therefore $\max(0, \delta-2) = \delta-2$. Let e be the exponent of G^{ab} . Then from the bound 1.6 we get

$$\begin{aligned}
|M(G)| &\leq p^{\frac{1}{2}d(n-k-e)+(\delta-1)k-\max(0,\delta-2)} \\
&= p^{\frac{1}{2}(d-1)(n-k-e)+\frac{1}{2}(n-k-e)+(\delta-1)k-(\delta-2)} \\
&= p^{\frac{1}{2}(d-1)(n-k-e)+\frac{1}{2}(n-k-e)+(\delta-1)(k-1)+1} \\
&= p^{\frac{1}{2}(d-1)(n-k)-\frac{1}{2}(d-1)e+\frac{1}{2}(n-k-e)+(d-1)(k-1)-(d-\delta)(k-1)+1} \\
&= p^{\frac{1}{2}(d-1)(n-k+2k-2)-\frac{1}{2}(d-1)e+\frac{1}{2}(n-k-e)-(d-\delta)(k-1)+1} \\
&= p^{\frac{1}{2}(d-1)(n+k-2)+1-[\frac{1}{2}(d-1)e-\frac{1}{2}(n-k-e)+(d-\delta)(k-1)]}
\end{aligned}$$

Now notice that $\frac{1}{2}(d-1)e - \frac{1}{2}(n-k-e)$ is a non-negative value. To see this, let $G^{ab} = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_d}}$ ($\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$). Then $n-k = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $e = \alpha_1$. Therefore

$$\begin{aligned}
\frac{1}{2}(d-1)e - \frac{1}{2}(n-k-e) &= \frac{1}{2}(d-1)\alpha_1 - \frac{1}{2}(\alpha_1 + \alpha_2 + \cdots + \alpha_d - \alpha_1) \\
&= \frac{1}{2} \left(\alpha_1 + \alpha_1 + \cdots + \alpha_1 \quad (d-1 \text{ times}) \right) - \frac{1}{2}(\alpha_2 + \cdots + \alpha_d) \\
&= \frac{1}{2}(\alpha_1 - \alpha_2) + (\alpha_1 - \alpha_3) + \cdots + (\alpha_1 - \alpha_d),
\end{aligned}$$

which is clearly a non-negative value because $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$. Obviously $(d-\delta)(k-1)$ is a non-negative value. Hence we have

$$|M(G)| \leq p^{\frac{1}{2}(d-1)(n+k-2)+1}.$$

The following Lemma sharpens the bound 1.7 for abelian p -groups.

Lemma 2.1. *Let G be an abelian p -group of order p^n such that $G = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_d}}$ ($\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$) and $|G| = p^n$, then $|M(G)| \leq p^{\frac{1}{2}(d(G)-1)(n-(\alpha_1-\alpha_d))}$.*

Proof. For $d = 2$, it is obvious from [6, Corollary 2.2.12]. So let us assume that $d \geq 3$. Let $\alpha_i = \frac{n-\alpha_1-\alpha_d}{d-2} + k_i$ for $i = 2, 3, \dots, d-1$. Then by [6, Corollary 2.2.12]

$$\begin{aligned}
|M(G)| &= p^{\alpha_2+2\alpha_3+\cdots+(d-1)\alpha_d} \\
&= p^{(d-1)(\alpha_2+\alpha_3+\cdots+\alpha_d)-(d-2)\alpha_2-(d-3)\alpha_3-\cdots-\alpha_{d-1}} \\
&= p^{(d-1)(n-\alpha_1)-(d-2)\alpha_2-(d-3)\alpha_3-\cdots-\alpha_{d-1}} \\
&= p^{(d-1)(n-\alpha_1)-\frac{(n-\alpha_1-\alpha_d)}{d-2}(1+2+\cdots+d-2)-\left[(d-2)k_2+(d-3)k_3+\cdots+k_{d-1}\right]} \\
&= p^{(d-1)(n-\alpha_1)-\frac{1}{2}(n-\alpha_1-\alpha_d)(d-1)-\left[(d-2)k_2+(d-3)k_3+\cdots+k_{d-1}\right]}.
\end{aligned}$$

Now observe that $k_2 + k_3 + \cdots + k_{d-1} = 0$. Also notice that there exist a j such that k_2, k_3, \dots, k_j are non-negative values and $k_{j+1}, k_{j+2}, \dots, k_{d-1}$ are non-positive values. It follows that the value $(d-2)k_2 + (d-3)k_3 + \cdots + k_{d-1}$ is non-negative. Therefore, $|M(G)| \leq p^{\frac{1}{2}(d(G)-1)(n-(\alpha_1-\alpha_d))}$. \square

Proof of Theorem 1.2 Let Ψ be as defined in [1, Proposition 1]. Following [1, Proposition 1] we have that

$$|M(G)| |\gamma_2(G)| |\text{Im } \Psi| \leq |M(G^{ab})| p^{k\delta}.$$

But note from [1, Proposition 1] that $|\text{Im}\Psi| \geq p^{\max(0, \delta-2)}$. Now the theorem follows from the Lemma 2.1.

Proof of Corollary 1.3 Having Theorem 1.2 in hand the proof of the corollary runs on the same lines as the proof of Theorem 1.1.

Proof of Theorem 1.4: By [11, Theorem 1.2] we have that

$$|\gamma_2(G)||M(G)| \leq |M(G/K)||G/\gamma_2(G)K \otimes K| \left| \frac{\gamma_2(G)}{\gamma_2(G) \cap K} \right|.$$

Since G is of nilpotency class at least 3, G/K is non-abelian. Hence $m \geq r + 2$. Applying Theorem 1.1 we get that

$$\begin{aligned} |\gamma_2(G)||M(G)| &\leq |G/\gamma_2(G)K \otimes K| p^{\frac{1}{2}(d(G/K)-1)(m+r-2)+1+r} \\ &= |G/\gamma_2(G)K \otimes K| p^{\frac{1}{2}d(G/K)(m+r-2)-\frac{1}{2}(m+r-2)+1+r} \\ &\leq |G/\gamma_2(G)K \otimes K| p^{\frac{1}{2}d(G/K)(m+r-2)-\frac{1}{2}(r+2+r-2)+1+r} \\ &\leq |G/\gamma_2(G)K \otimes K| p^{\frac{1}{2}d(G/K)(m+r-2)+1}. \end{aligned}$$

This proves the theorem.

Proof of Theorem 1.5: Let c be the nilpotency class of G . It is obvious that $c \leq k+1$ and $d(G) \leq n-k$. Therefore $d(G) \leq n-c+1 = r+1$, so that $d(G)-1 \leq r$. The inequality $c \leq k+1$ can be written as $n-(n-c) \leq k+1$, i.e., $n-r \leq k+1$ so that $n+k-2 \leq r+2k-1$. Using Theorem 1.1 with the inequalities $d(G)-1 \leq r$ and $n+k-2 \leq r+2k-1$ we get the required result.

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